Relativistic plasma viscosity of the Burnett kind

V. S. Tsypin,[1] R. M. O. Galvão,[1] I. C. Nascimento,[1] N. L. Tsintsadze,[2] L. N. Tsintsadze,[2] M. Tendler,[3] and J. P. Neto[4]
[1]*Physics Institute, University of São Paulo, Cidade Universitaria 05508-900, Sào Paulo, Brazil*
[2]*Institute of Physics, Academy of Sciences, Street Tamarashvili 6, Tbilisi-380077, Georgia*
[3]*The Alfvén Laboratory, Royal Institute of Technology, 10044 Stockholm, Sweden*
[4]*National Laboratory of Scientific Computations, Petrópolis, Rio de Janeiro, Brazil*
(Received 26 March 1999)

Hydrodynamic equations to describe relativistic and ultrarelativistic plasma dynamics were obtained by Dzhavakhishvili and Tsintsadze [Sov. Phys. JETP **37**, 666 (1973)] using the Chapman and Enskog scheme to solve the relativistic kinetic equations for the different plasma species. This approach leads to a representation of the particle viscosities in the Navier-Stokes form and, therefore, some relevant physical processes, such as the Burnett type of particle viscosity, cannot be properly dealt with in this scheme. In this paper we employ the extended Grad method to derive hydrodynamic equations which include ultrarelativistic viscosities of the Burnett type, i.e., viscosities that depend not only on derivatives of the particle macroscopic velocities but also on derivatives of particle heat fluxes. [S1063-651X(99)04010-6]

PACS number(s): 52.60.+h, 52.25.Fi, 52.30.−q

I. INTRODUCTION

The hydrodynamic description of plasma dynamics has some advantages in comparison with the kinetic one. In particular, the magnetohydrodynamic (MHD) equations are more appropriate to follow the dynamics of macroscopic plasma quantities, such as the plasma density, velocity, and temperature in complicated physical systems. The usefulness of the hydrodynamic approach has been put into evidence on many occasions; for instance, the hydrodynamic equations of Braginskii [1] have been widely employed for laboratory and space plasmas for a long time. However, because the Braginskii equations are valid only for nonrelativistic plasmas, there are cosmic plasmas which cannot be described by these equations, e.g., electron-positron plasmas [2,3], dense cosmic rays, and strong-current relativistic electron beams. For these plasmas, the particle velocities $\mathbf{V}_a$ (''a'' is the particle kind) are of the order of the speed of light c and the particle temperatures T_a are about the electron or positron rest energies $\varepsilon_{e0}=M_e c^2$, i.e., $T_a \geqslant \varepsilon_{e0}$. Hydrodynamic equations for relativistic and ultrarelativistic plasmas were obtained in Ref. [4]; the relativistic kinetic equations for the different plasma species were solved by means of the Chapman and Enskog procedure [5], also used by Braginskii [1]. In this scheme, the particle distribution function is expanded in a power series on the small parameters of the problem such as the Knudsen number $\mathrm{Kn}_a=\lambda_a/L_p$ and $1/\nu_{ab}t_{\mathrm{ch}}$. Here, λ_a is the mean free path of plasma species ''a,'' L_p is the characteristic length of the macroscopic plasma quantity profiles, ν_{ab} is the collision frequency of species ''a'' and ''b,'' and t_{ch} is the characteristic time for variation of macroscopic plasma quantities.

The MHD equations for ultrarelativistic plasmas derived by Dzhavakhishvili and Tsintsadze [4] take into account interesting effects which are absent in the nonrelativistic equations of Braginskii [1]. We mention, for example, the dependence of particle heat fluxes not only on temperature gradients but also on electric fields, particle density gradients, and macroscopic velocities. So far, these effects have not been fully explored. Nevertheless, there are problems on plasma dynamics which cannot be properly dealt with by Braginskii [1] and Dzhavakhishvili and Tsintsadze [4] equations because these equations keep the viscosity in the Navier-Stokes approximation, which contains only derivatives of the macroscopic plasma velocity.

There are some important physical processes on which we would like to focus our attention, namely, the so-called thermostress gas [6] and plasma [7] flows connected with the dependence of the gas or plasma viscosities on the particle heat fluxes (the viscosities of the Burnett kind [8]). The magnitude of these flows is of the level of the so-called drift velocities $V_a^{\mathrm{dr}} \approx \rho_a v_{Ta}/L_p$, where $\rho_a = v_{Ta}/\omega_{ca}^n$ is the particle Larmour radius, $v_{Ta}=\sqrt{2T_a/M_a}$ is the particle thermal velocity, $\omega_{ca}^n = e_a B/M_a c$ is the nonrelativistic particle cyclotron frequency, e_a and M_a are the particle charge and mass, respectively, and B is the magnetic field. Usually, it is assumed that parameter ρ_a/L_p is small, i.e., $\rho_a/L_p \ll 1$, so that the drift flows are small, $V_a^{\mathrm{dr}} \ll v_{Ta}$. However, under some conditions, these slow flows can be the only flows in both collisional and weakly collisional plasmas, as we show in the sequel. These plasma flows cannot be investigated on the basis of the Navier-Stokes equations. In the absence of external forces, momentum sources, and boundary flows in confined plasmas, all plasma flows damp due to viscosity in the Navier-Stokes scheme. On the contrary, under the same conditions, steady plasma flows induced by particle temperature gradients are allowed in the Burnett equations. For collisional plasmas, this kind of nonrelativistic plasma viscosity was obtained in Refs. [9] and [10] and is widely used to investigate flows in plasmas [7,11]. The so-called residual plasma rotation in tokamaks is connected with the dependence of the ion viscosity on derivatives of the ion heat flux, both in collisional [11,12] and weakly collisional plasmas[13]. There exists also a kind of plasma drift instability, with the frequency ω satisfying the inequality $V_a^{\mathrm{dr}}/L_p \approx \rho_a v_{Ta}/L_p^2 < \omega \ll \omega_{ca}^n$, which should be investigated on the basis of Burnett transport equations [9]. Consequently, it may also be important to obtain the Burnett kind of plasma viscosities for relativistic and, specifically, ultrarelativistic plasmas.

In the Chapman and Enskog scheme, employing an expansion of the particle distribution function in a power series on the Knudsen number Kn_a, which was used to obtain the plasma viscosities in Refs. [1] and [4], it is very cumbersome to get the Burnett kind of viscosity. This difficulty is connected with the necessity to expand the distribution function up to the second order on the parameters Kn_a or ρ_a/L_p. For this reason, in Refs. [9], [10], and [12] the Grad method [14] was employed to obtain more complete MHD equations. In this paper, we include ultrarelativistic plasmas characterized by $T_a \geqslant \varepsilon_{e0}$, under the condition that the mean particle velocities $\mathbf{V}_a$ are much smaller than the speed of light c, i.e., $V_a < c$. In the extended Grad approximation [9,10,12], all moments of the distribution function, such as the particle density n_a, velocity $\mathbf{V}_a$, temperature T_a, viscosity π_a, heat flux $\mathbf{q}_a$, and so on, are considered to be equally important. This is in contrast with the Chapman and Enskog scheme, where $n_a, \mathbf{V}_a, T_a$ are considered as the main moments and all other moments are functions of them. The Grad method substantially simplifies the calculations. The resulting expressions coincide with the transport equations obtained by Dzhavakhishvili and Tsintsadze [4], except for the plasma viscosity. As an example, we find the viscosity of rarefied electron-positron plasmas in the case when the electron-positron inelastic cross sections are much smaller than the elastic collision cross section.

II. BASIC EQUATIONS

As usual, to obtain the MHD equations we proceed from the relativistic plasma kinetic equations [4,15] for the particle distribution functions $f_a(t,\mathbf{r},\mathbf{p}_a)$ in the presence of electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, respectively,

$$\frac{\partial f_a}{\partial t}+\frac{c^2}{\varepsilon_a}\mathbf{p}_a\frac{\partial f_a}{\partial \mathbf{r}}+e_a\left\{\mathbf{E}+\frac{c}{\varepsilon_a}[\mathbf{p}_a\times\mathbf{B}]\right\}\frac{\partial f_a}{\partial p_a}=\sum_b C_{ab}(f_a,f_b), \qquad (1)$$

where $\varepsilon_a=c(p_a^2+M_a^2c^2)^{1/2}$ and $C_{ab}(f_a,f_b)$ is a collision term, which will be defined below. As in Ref. [4], in the rest frame of a given plasma component, we introduce the definitions

$$\int f_a d\mathbf{p}_a=n_a, \quad c^2\int\frac{\mathbf{p}_a}{\varepsilon_a}f_a d\mathbf{p}_a=0, \quad G(z_a)=\frac{K_3(z_a)}{K_2(z_a)},$$
$$\int(\varepsilon_a-M_ac^2)f_a d\mathbf{p}_a=n_a[M_ac^2(G_a-1)-T_a], \qquad (2)$$
$$z_a=\frac{M_ac^2}{T_a},$$

where $K_n(z_a)$ is the Macdonald function of the nth order.

Multiplying Eq. (1) by 1, $\mathbf{p}_a$, and $\varepsilon_a-M_ac^2$, and integrating over the momentum space, we obtain the equations of continuity, momentum, and the thermal balance, respectively, for the macroscopic particle density n_a, mean particle velocity $\mathbf{V}_a$, and temperature T_a,

$$\frac{\partial}{\partial t}(\gamma_a n_a)+\nabla\cdot(\gamma_a n_a\mathbf{V}_a)=\mathbf{0}, \qquad (3)$$

$$\begin{aligned}\gamma_a n_a\frac{d_a}{dt}(\gamma_a M_a G_a V_{ai})=&-\frac{\partial P_a}{\partial x_i}-\frac{\partial}{\partial x_k}(s_{aim}s_{akn}\pi_{amn})+\gamma_a n_a e_a\left\{\mathbf{E}+\frac{1}{c}[\mathbf{V}_a\times\mathbf{B}]\right\}_i\\&+s_{aik}R_{ak}+\frac{1}{c^2}\gamma_a V_{ai}Q_a-\frac{1}{c^2}\frac{\partial}{\partial t}\left[\gamma_a s_{aik}V_{am}\pi_{akm}+\gamma_a\left(s_{aik}+\frac{1}{c^2}\gamma_a V_{ai}V_{ak}\right)q_{ak}\right]\\&-\frac{1}{c^2}\frac{\partial}{\partial x_k}[\gamma_a(s_{aim}V_{ak}+s_{akm}V_{ai})q_{am}],\end{aligned} \qquad (4)$$

$$\begin{aligned}n_a\frac{d_a}{dt}(M_ac^2G_a-T_a)-T_a\frac{d_an_a}{dt}=&-\frac{\partial}{\partial x_k}(\gamma_a^{-1}s_{akm}q_{am})-s_{aim}s_{akn}\pi_{amn}\frac{\partial V_{ai}}{\partial x_k}+\gamma_a^{-1}Q_a-\frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{q}_a\cdot\mathbf{V}_a)\\&-\frac{1}{c^2}\left[\gamma_a s_{aik}V_{am}\pi_{akm}+\gamma_a\left(s_{aik}+\frac{1}{c^2}\gamma_a V_{ai}V_{ak}\right)q_{ak}\right]\frac{\partial V_{ai}}{\partial t}\\&-\frac{1}{c^2}\gamma_a(s_{aim}V_{ak}+s_{akm}V_{ai})q_{am}\frac{\partial V_{ai}}{\partial x_k}.\end{aligned} \qquad (5)$$

Here,

$$s_{aik}=\delta_{ik}+(\gamma_a-1)\frac{V_{ai}V_{ak}}{V_a^2}, \quad \gamma_a=\left(1-\frac{V_a^2}{c^2}\right)^{-1/2},$$

$$\frac{d_a}{dt}=\frac{\partial}{\partial t}+\mathbf{V}_a\cdot\nabla,$$

$P_a=n_aT_a$ is the particle pressure, π_{aik} is the viscous tensor,

$\mathbf{q}_a$ is the heat flux, and $\mathbf{R}_a$ and Q_a are the friction force and the collisional heat release, respectively. These quantities are defined by the expressions [4]

$$P_a=\frac{c^2}{3}\int\frac{p_a^2}{\varepsilon_a}f_a d\mathbf{p}_a,$$

$$\pi_{aik}=c^2\int\frac{1}{\varepsilon_a}\left(p_{ai}p_{ak}-\frac{p_a^2}{3}\delta_{ik}\right)f_a d\mathbf{p}_a, \tag{6}$$

$$\mathbf{q}_a=c^2\int\mathbf{p}_a f_a d\mathbf{p}_a,$$

$$\mathbf{R}_a=\int\mathbf{p}_a C_a d\mathbf{p}_a,\quad Q_a=(\varepsilon_a-M_a c^2)C_a d\mathbf{p}_a.$$

In comparison with Ref. [4], we have corrected some misprints in Eq. (5), namely the parameter γ_a appears correctly with power -1 in the first and third terms on the right-hand side of Eq. (5).

III. ULTRARELATIVISITC PLASMA

As we confine ourselves to the ultrarelativistic case, $z_a \ll 1$ and $V_a < c$, in the reference frame, where the mean velocity of the particle kind "a" is equal to 0, the equilibrium distribution functions can be chosen in the form [4,15]

$$f_a^{(0)}=\frac{n_a}{8\pi}\left(\frac{c}{T_a}\right)^3\exp\left(-\frac{pc}{T_a}\right),$$

$$f_b^{(0)}=\frac{n_b}{8\pi}\left(\frac{c}{T_b}\right)^3\exp\left\{-\frac{1}{T_b}[pc-\mathbf{p}\cdot(\mathbf{V}_b-\mathbf{V}_a)]\right\}. \tag{7}$$

In the following, we assume that the quasineutrality condition $n_a\approx n_b=n_0$ is fulfilled and that the particle temperatures are approximately equal, $T_a\approx T_b=T_0$. In this reference frame, the kinetic equation (1) can be approximately written as [4]

$$\sum_b C_{ab}(f_a,f_b)-\frac{cM_a^*}{p}[\mathbf{p}\times\boldsymbol{\omega}_{ca}]\frac{\partial f_a}{\partial\mathbf{p}}=\frac{d_a f_a}{dt}+\frac{c}{p}\mathbf{p}\frac{\partial f_a}{\partial\mathbf{r}}+\left(e_a\mathbf{E}^*-\frac{p}{c}\frac{d_a\mathbf{V}_a}{dt}\right)\frac{\partial f_a}{\partial\mathbf{p}}-\frac{\partial V_{ai}}{\partial x_k}p_k\frac{\partial f_a}{\partial p_i}, \tag{8}$$

where

$$\boldsymbol{\omega}_{ca}=\frac{e_a\mathbf{B}}{cM_a^*},\quad \mathbf{E}^*=\mathbf{E}+\frac{1}{c}[\mathbf{V}_a\times\mathbf{B}],\quad M_a^*=\frac{4T_0}{c^2}\gg M_a.$$

In both the Chapman and Enskog [5] and Grad [14] schemes, the distribution function is supposed to be close to a Maxwellian,

$$f_a=f_a^{(0)}(1+\Phi_a),\quad \Phi_a\ll 1. \tag{9}$$

According to the extended Grad scheme, we write

$$\Phi_a=\Phi_{a0}+p_i\Phi_{ai}+(p_ip_k-\tfrac{1}{3}p^2\delta_{ik})\Phi_{aik}+\cdots. \tag{10}$$

Here, Φ_{a0} is the scalar part, which can depend on scalar moments of the distribution function but n_0,T_0; Φ_{ai} is the vector part depending on vector moments except $\mathbf{V}_a$; Φ_{aik} is the tensor part depending on π_{aik} and other tensor moments of the second rank. The scalar part Φ_{a0} and the highest-order tensor moments are not used in this paper. Thus, we obtain equations for tensor moments of the second rank. We remind the reader that it is appropriate to substitute the zero approximation distribution function Eq. (7) in the right-hand side of Eq. (8) to find equations for the moments of the distribution function in the Chapman and Enskog scheme [5]. In the Grad scheme [9,14], one uses the distribution function Eq. (9) to substitute both parts of Eq. (8). The small parameters that we consider are the Knudsen number Kn_a and the ratio of the particle Larmour radius ρ_a to the characteristic length L_p of the spatial variation of the macroscopic plasma quantities, ρ_a/L_p.

To find the moments of the distribution function, we multiply Eq. (8) by a proper function X of the momentum $\mathbf{p}$ and then integrate over the momentum space,

$$\begin{aligned}\sum_b\langle XC_{ab}(f_a,f_b)\rangle&+cM_a^*\left\langle\left[\frac{\mathbf{p}}{p}\times\boldsymbol{\omega}_{ca}\right]\frac{\partial X}{\partial\mathbf{p}}\right\rangle\\&=\frac{d_a}{dt}\langle X\rangle-\left\langle\frac{d_aX}{dt}\right\rangle+c\frac{\partial}{\partial\mathbf{r}}\left\langle\frac{\mathbf{p}}{p}X\right\rangle-c\left\langle\frac{\mathbf{p}}{p}\frac{\partial}{\partial\mathbf{r}}X\right\rangle+\left\langle X\left(\frac{\mathbf{p}}{cp}\frac{d_a\mathbf{V}_a}{dt}+\boldsymbol{\nabla}\times\mathbf{V}_a\right)\right\rangle\\&-\left\langle\left\{e_a\mathbf{E}^*-\frac{p}{c}\frac{d_a\mathbf{V}_a}{dt}-(\mathbf{p}\times\boldsymbol{\nabla})\mathbf{V}_a\right\}\frac{\partial X}{\partial\mathbf{P}}\right\rangle,\end{aligned} \tag{11}$$

where $\langle X\rangle=\int f_a X\,d\mathbf{p}$ and X is the proper scalar, vector, or tensor function of the momentum $\mathbf{p}$.

IV. PARTICLE VISCOSITIES

According to Refs. [4], [9], [10], and [12], we can write

$$\Phi_{aik}=\sum_{l=0}^{\infty} a_{aik}^{(l)}L_l^{(5)}(y_a), \quad y_a=\frac{pc}{T_0}, \tag{12}$$

where $L_l^{(5)}(y_a)$ are the Sonine-Laguerre polynomials, and keep only two terms in the expansion in Eq. (10). The errors in the transport coefficients due to this approximation are not larger than a few percent of the exact values, and taking into account three or more terms in Eq. (10) greatly increases the algebra without substantially improving the accuracy. Using the definitions given by Eq. (6), we obtain

$$a_{aik}^{(0)}=\frac{c^2}{8n_0T_0^3}\pi_{aik}, \quad a_{aik}^{(1)}=\frac{c^2}{8n_0T_0^3}\pi_{aik}^*. \tag{13}$$

As in Refs. [9], [10], and [12], we introduce the tensor π_{aik}^* in Eq. (13), which is defined by the relation $\pi_{aik}^*=(c/6)\int[p_{ai}p_{ak}-(p_a^2/3)\delta_{ik}]L_1^{(5)}(y_a)f_a d\mathbf{p}_a/p_a$. Integrating Eq. (11) with the weights $X=(p_ip_k-p^2\delta_{ik}/3)L_0^{(5)}(y_a)$ and $X=(p_ip_k-p^2\delta_{ik}/3)L_1^{(5)}(y_a)$, we obtain the equations for the particle viscosity π_{aik} and its analog π_{aik}^*,

$$4\omega_{ca}\hat{\sigma}\pi_{aik}+\frac{c^2}{T_0}\sum_b\int(p_ip_k-\tfrac{1}{3}p^2\delta_{ik})L_0^{(5)}(y_a)C_{ab}(f_a,f_b)d\mathbf{p}_a=4n_0T_0W_{aik}^{(1)} \tag{14}$$

and

$$4\omega_{ca}\hat{\sigma}\pi_{aik}^*+\frac{c^2}{6T_0}\sum_b\int(p_ip_k-\tfrac{1}{3}p^2\delta_{ik})L_1^{(5)}(y_a)C_{ab}(f_a,f_b)d\mathbf{p}_a=n_0T_0W_{aik}^{(2)}, \tag{15}$$

respectively. Here,

$$W_{aik}^{(1)}=\langle\boldsymbol{\nabla}\cdot\mathbf{V}_a\rangle_{ik}+\frac{1}{4n_0T_0^2}\langle\boldsymbol{\nabla}\cdot[T_0(2\mathbf{q}_a-\mathbf{q}_a^*)]\rangle_{ik}, \tag{16}$$

$$W_{aik}^{(2)}=\frac{1}{n_0T_0^2}\langle\boldsymbol{\nabla}\cdot[T_0(2\mathbf{q}_a^*-\mathbf{q}_a)]\rangle_{ik}+\frac{1}{3n_0T_0^2}\langle\boldsymbol{\nabla}T_0\cdot(2\mathbf{q}_a-\mathbf{q}_a^*)\rangle_{ik}, \tag{17}$$

$$\langle\mathbf{A}\cdot\mathbf{B}\rangle_{ik}=A_iB_k+A_kB_i-\tfrac{2}{3}\delta_{ik}\mathbf{A}\cdot\mathbf{B},$$

$$\hat{\sigma}\pi_{ik}=h_\tau\epsilon_{\tau\lambda\mu}(\pi_{i\mu}\delta_{k\lambda}+\pi_{k\mu}\delta_{i\lambda}), \quad h_\tau=\frac{B_\tau}{B}. \tag{18}$$

The moment $\mathbf{q}_a^*$ is an analog of the heat flux $\mathbf{q}_a$, which is defined below (Sec. V) and B_τ is the τ component of the magnetic field $\mathbf{B}$.

For the case when both kinds of particles "a" and "b" are ultrarelativistic, the integrals in Eqs. (14) and (15) can be transformed to the form

$$\sum_b\int X(p_a)C_{ab}(f_a,f_b)d\mathbf{p}_a=\sum_b\frac{3n_a\nu_{ab}}{4\pi}\int d\mathbf{y}_aX(y_a)\exp(-y_a)\times\left\{\delta_{ik}\frac{\partial^2\Phi_a(y_a)}{\partial y_{ai}\partial y_{ak}}-\frac{y_{ak}}{y_a}\frac{\partial\Phi_a(y_a)}{\partial y_{ak}}\right\}. \tag{19}$$

Using the orthogonality property of the Sonine-Laguerre polynomials [16]

$$\int_0^\infty y^ke^{-y}L_m^k(y)L_n^k(y)dy=\frac{(m+k)!}{m!}\delta_{mn},$$

we find from Eqs. (14), (15), and (19)

$$\omega_{ca}\hat{\sigma}\pi_{aik}-3\nu_a\pi_{aik}=P_aW_{aik}^{(1)}, \tag{20}$$

$$4\omega_{ca}\hat{\sigma}\pi_{aik}^*-18\nu_a\pi_{aik}^*=P_aW_{aik}^{(2)}, \tag{21}$$

where

$$\nu_{ab}=\frac{\pi e_a^2e_b^2Ln_0c}{3T_0^2}, \quad \nu_a\equiv\nu_{aa}, \quad \omega_{ca}=\frac{e_aBc}{4T_0}. \tag{22}$$

$L=\ln(b_{\max}/b_{\min})$ is the Coulomb logarithm defined in Ref. [4]. Here, $b_{\max}$ and $b_{\min}$ are the characteristic maximal and minimal collision parameters, respectively. The maximum impact parameter $b_{\max}$ for ultrarelativistic plasmas is chosen to be the Debye screening radius $r_D=(T_0/4\pi n_0e_a^2)^{1/2}$ with the factor $e_a^2/hc=1/137$, i.e., $b_{\max}=r_De_a^2/hc$ [17]. The lower impact parameter $b_{\min}$ is equal to $b_{\min}=e_a^2/2T_0$, i.e., the value at which a deviation by an angle $\sim\pi/2$ takes place [17]. Using conditions $b_{\max}>b_{\min}$ and $T_0>M_ac^2$, we find the critical particle density $n_{\rm cr}$ at which elastic collisions take place, $n_{\rm cr}<(T_0^3/\pi e_a^6)(e_a^4/h^2c^2)$. From another side, the particle mean free path $\lambda_a=c/\nu_a$ should be less than the characteristic scale of the system, L_p. From this condition, we find the lower limit for the critical density, $n_{\rm cr}>3T_0^2/(e_a^4L_pL)$. We remark that both inequalities are in no contradiction. In Eqs. (20) and (21), there is only the collision frequency ν_a, as we consider in the sequel only the electron-positron case for which the equality $e_a^2=e_b^2$ is fulfilled.

In contrast to the Braginskii [1] and Dzhavakhishvili and Tsintsadze [4] approximations, we see from Eqs. (16)–(21) that the particle viscosities depend on derivatives of particle heat fluxes and their analogs. In contrast to the nonrelativistic case [1,9,10], Eqs. (20) and (21) are not coupled. Thus, solutions of Eqs. (16)–(21) can be found independently and they are [1,4]

$$\pi_{aik}=-\eta_{0a}W_{a0ik}^{(1)}-\eta_{1a}W_{a1ik}^{(1)}-\eta_{2a}W_{a2ik}^{(1)}+\eta_{3a}W_{a3ik}^{(1)}+\eta_{4a}W_{a4ik}^{(1)}, \tag{23}$$

$$\pi_{aik}^{*}=-\eta_{0a}^{*}W_{a0ik}^{(2)}-\eta_{1a}^{*}W_{a1ik}^{(2)}-\eta_{2a}^{*}W_{a2ik}^{(2)}+\eta_{3a}^{*}W_{a3ik}^{(2)}+\eta_{4a}^{*}W_{a4ik}^{(2)}, \tag{24}$$

where

$$W_{a0ik}^{(1,2)}=\tfrac{3}{2}(h_ih_k-\tfrac{1}{3}\delta_{ik})(h_\mu h_\nu-\tfrac{1}{3}\delta_{\mu\nu})W_{\mu\nu}^{(1,2)}, \tag{25}$$

$$W_{a1ik}^{(1,2)}=[(\delta_{i\mu}-h_ih_\mu)(\delta_{k\nu}-h_kh_\nu)+\tfrac{1}{2}(\delta_{ik}-h_ih_k)h_\mu h_\nu]W_{\mu\nu}^{(1,2)}, \tag{26}$$

$$W_{a2ik}^{(1,2)}=[(\delta_{i\mu}-h_ih_\mu)h_kh_\nu+(\delta_{k\nu}-h_kh_\nu)h_ih_\mu]W_{\mu\nu}^{(1,2)}, \tag{27}$$

$$W_{a3ik}^{(1,2)}=\tfrac{1}{2}[(\delta_{i\mu}-h_ih_\mu)\epsilon_{k\gamma\nu}+(\delta_{k\nu}-h_kh_\nu)\epsilon_{i\gamma\mu}]h_\gamma W_{\mu\nu}^{(1,2)}, \tag{28}$$

$$W_{a4ik}^{(1,2)}=(h_ih_\mu\epsilon_{k\gamma\nu}+h_kh_\nu\epsilon_{i\gamma\mu})h_\gamma W_{\mu\nu}^{(1,2)}. \tag{29}$$

The viscous coefficients η can be found after the substitution of Eqs. (23) and (24) into Eqs. (20) and (21),

$$\eta_{0a}=\frac{P_a}{3\nu_a},\quad \eta_{1a}=\frac{3P_a}{4\nu_a\Delta_{1a}},\quad \eta_{2a}=\frac{3P_a}{\nu_a\Delta_{2a}},\quad \eta_{3a}=\frac{x_aP_a}{2\nu_a\Delta_{1a}},\quad \eta_{4a}=\frac{x_aP_a}{\nu_a\Delta_{2a}}, \tag{30}$$

$$\Delta_{1a}=x_a^2+\tfrac{9}{4},\quad \Delta_{2a}=x_a^2+9,\quad x_a=\frac{\omega_{ca}}{\nu_a}, \tag{31}$$

$$\eta_{0a}^{*}=\frac{P_a}{18\nu_a},\quad \eta_{1a}^{*}=\frac{9P_a}{32\nu_a\Delta_{1a}^{*}},\quad \eta_{2a}^{*}=\frac{9P_a}{8\nu_a\Delta_{2a}^{*}},\quad \nu_{3a}^{*}=\frac{x_aP_a}{8\nu_a\Delta_{1a}^{*}},\quad \eta_{4a}^{*}=\frac{x_aP_a}{4\nu_a\Delta_{2a}^{*}}, \tag{32}$$

$$\Delta_{1a}^{*}=x_a^2+\tfrac{81}{16},\quad \Delta_{2a}^{*}=x_a^2+\tfrac{9}{4}. \tag{33}$$

To find the coefficients of Eqs. (23) and (24), given by Eqs. (30) and (31), the relations

$$W_{\mu\nu}^{(1,2)}=W_{a0ik}^{(1,2)}+W_{a1ik}^{(1,2)}+W_{a2ik}^{(1,2)},\quad \hat{\sigma}W_{a0ik}^{(1,2)}=0,\quad W_{apik}^{(1,2)}W_{aqik}^{(1,2)}=0\quad \text{if}\quad p\neq q, \tag{34}$$

$$\hat{\sigma}W_{a1ik}^{(1,2)}=-2W_{a3ik}^{(1,2)},\quad \hat{\sigma}W_{a3ik}^{(1,2)}=2W_{a1ik}^{(1,2)},\quad \hat{\sigma}W_{a2ik}^{(1,2)}=-W_{a4ik}^{(1,2)},\quad \hat{\sigma}W_{a4ik}^{(1,2)}=W_{a2ik}^{(1,2)}. \tag{35}$$

were used to solve Eqs. (20) and (21).

V. HEAT FLUXES

The heat fluxes are connected to the coefficients $a_i^{(l)}$ in the expansion

$$\Phi_{ai}=\sum_{l=1}^{\infty} a_{ai}^{(l)}L_l^{(3)}(y_a) \tag{36}$$

through the relations

$$a_{ai}^{(1)}=-\frac{q_{ai}}{4n_0T_0^2},\quad a_{ai}^{(2)}=-\frac{q_{ai}^{*}}{4n_0T_0^2}, \tag{37}$$

where the heat flux analog is defined by the relation $\mathbf{q}_a^{*}=(c^2/5)\int\mathbf{P}_a[6+L_1^{(3)}(y_a)]f_ad\mathbf{P}_a$. The equations for the particle heat fluxes can be obtained integrating Eq. (11) with the weights $X=p_iL_1^{(3)}(y_a)$ and $X=p_iL_2^{(3)}(y_a)$,

$$\frac{4}{c^2}\omega_{ca}\epsilon_{ikm}q_{ak}h_m-\sum_b\int p_iL_1^{(3)}(y_a)C_{ab}(f_a,f_b)d\mathbf{p}_a=4n_0\boldsymbol{\nabla}_iT_0, \tag{38}$$

$$\frac{10}{c^2}\frac{M_a}{T_0}\omega_{ca}\epsilon_{ikm}q_{ak}^{*}h_m-\sum_b\int p_iL_2^{(3)}(y_a)C_{ab}(f_a,f_b)d\mathbf{p}_a=0. \tag{39}$$

Using the orthogonality property of the Sonine-Laguerre polynomials, we find from Eqs. (38) and (39),

$$\omega_{ca}\epsilon_{ikm}q_{ak}h_m-6\nu_aq_{ai}=\frac{c^2P_a}{T_0}\boldsymbol{\nabla}_iT_0, \tag{40}$$

$$\omega_{ca}\epsilon_{ikm}q_{ak}^{*}h_m-9\nu_aq_{ai}^{*}=0. \tag{41}$$

Again, in contrast to the nonrelativistic case [1,9,10], Eqs. (40) and (41) are not coupled. We obtain the expressions for $\mathbf{q}_a$ and $\mathbf{q}_a^{*}$ from Eqs. (40) and (41),

$$\mathbf{q}_a=-\kappa_{a\parallel}\mathbf{h}\boldsymbol{\nabla}_{\parallel}T_0+\kappa_{a\wedge}[\mathbf{h}\times\boldsymbol{\nabla}T_0]-\kappa_{a\perp}[\mathbf{h}\times[\boldsymbol{\nabla}T_0\times\mathbf{h}]], \tag{42}$$

$$\mathbf{q}_a^{*}=\mathbf{0}, \tag{43}$$

where the heat conductivities are given by the expressions

$$\kappa_{a\parallel}=\frac{c^2P_a}{6\nu_aT_0},\quad \kappa_{a\wedge}=\frac{2x_ac^2P_a}{\nu_aT_0\Delta_a},\quad \kappa_{a\perp}=\frac{6c^2P_a}{\nu_aT_0\Delta_a},\quad \Delta_a=x_a^2+36. \tag{44}$$

Substituting Eqs. (42) and (43) into Eqs. (16) and (17), we find from Eq. (23) the Burnett kind of ultrarelativistic plasma viscosity. We see that in contrast with Refs. [1] and [4], in addition to the derivatives of the macroscopic plasma velocities V_a, Eq. (23) contains terms connected with derivatives of T_0. Similar equations were obtained for the nonrelativistic case in both gases [6] and plasmas [9,10].

VI. CONCLUSION

Transport equations with the Burnett kind of particle viscosities for ultrarelativistic plasmas are derived in this paper. This kind of plasma viscosities differs from the Navier-Stokes viscosities because of their dependence on the derivatives of the particle heat fluxes in addition to the conventional dependence on the derivatives of the particle velocities. Instead of using the conventional Chapman and Enskog scheme, which makes it very difficult to derive equations including this type of viscosity, for the reason explained in the Introduction, the extended Grad method is adopted in this paper. In this method all moments of the distribution function are considered to be equally important, and the expansion procedure substantially facilitates the derivation of ultrarelativistic particle viscosities of the Burnett kind. As an example, the viscosity of rarefied electron-positron plasmas, in the case when the electron-positron inelastic cross section is much smaller than the elastic collision cross section, is derived in the paper.

ACKNOWLEDGMENTS

The work was supported by FAPESP (São Paulo Foundation for Research Support) and PRONEX (Superior Scientific Projects) RMOG 050/97 grant from the Ministry of Science and Technology, Brazil.

[1] S. I. Braginskii, in *Review of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[2] Ya. B. Zeldovich and I. D. Novikov, in *Relativistic Astrophysics*, edited by K. S. Thorne and W. D. Arnett (The University of Chicago Press, Chicago, 1971), Vols. 1 and 2.
[3] P. K. Shukla, N. N. Rao, My Yu, and N. L. Tsintsadze, Phys. Rep. **138**, 1 (1986).
[4] D. I. Dzhavakhishvili and N. L. Tsintsadze, Zh. Eksp. Teor. Fiz. **64**, 1314 (1973) [Sov. Phys. JETP **37**, 666 (1973)].
[5] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-uniform Gases* (Cambridge University Press, New York, 1960).
[6] M. N. Kogan, V. Galkin, and O. G. Fridlender, Usp. Fiz. Nauk **119**, 111 (1976) [Sov. Phys. Usp. **19**, 420 (1976)].
[7] S. M. Dikman and L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **78**, 1752 (1980) [Sov. Phys. JETP **51**, 879 (1980)].
[8] D. Burnett, Proc. London Math. Soc. **39**, 385; **40**, 382 (1935).
[9] A. B. Mikhailovskii and V. S. Tsypin, Plasma Phys. **13**, 785 (1971).
[10] A. B. Mikhailovskii and V. S. Tsypin, Beitr. Plasmaphys. **24**, 317 (1984).
[11] R. D. Hazeltine, Phys. Fluids **17**, 961 (1974).
[12] A. B. Mikhailovskii and V. S. Tsypin, Zh. Eksp. Teor. Fiz. **83**, 139 (1982) [Sov. Phys. JETP **56**, 75 (1982)].
[13] F. L. Hinton and M. N. Rosenbluth, Phys. Fluids **16**, 836 (1973).
[14] H. Grad, Commun. Pure Appl. Math. **2**, 331 (1949).
[15] S. T. Belyaev and G. I. Budker, Dokl. Akad. Nauk **106**, 807 (1956) [Sov. Phys. Dokl. **1**, 218 (1956)].
[16] M. Abramovitz and I. A. Stegan, in *Handbook of Mathematical Functions* (Dover, New York, 1972).
[17] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).